Exact Travelling Wave Solutions of Some Nonlinear Nonlocal Evolutionary Equations

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Abstract. Direct algebraic method of obtaining exact solutions to nonlinear PDE's is applied to certain set of nonlinear nonlocal evolutionary equations, including nonlinear telegraph equation, hyperbolic generalization of Burgers equation and some spatially nonlocal hydrodynamic-type model. Special attention is paid to the construction of the kink-like and soliton-like solutions.

1 Introduction

In recent decades the problem of obtaining exact solutions of nonlinear evolutionary equations has attracted attention of many experts in mathematical physics. The most fundamental achievement in this area was development of the inverse scattering method [1]. Unfortunately, this method is applicable to the relatively narrow class of the completely integrable equations, while for the majority of nonlinear evolutionary equations methods of obtaining the general solution do not exist. Yet even the possibility of obtaining particular solutions to the nonlinear PDE's very often is considered in applications as a big success, because having the analytical solutions of a modelling system it is more easy to analyze it and to make its interpretation. Exact solutions are also widely used as a starting point for various asymptotic methods, for testing the numerical schemes and facilitating the stability analysis.

For nonlinear PDE's, which are not integrable, exact solutions are usually obtained by means of the Group Theory Reduction [2]. Appreciating all advantages of the symmetry-based methods, we would like to pay attention to the fact that they are not fully universal, firstly, because they can be effectively used merely in case the PDE's possess some non-trivial symmetry, and, secondly, because it is very difficult within the frameworks of these methods to obtain any solution with the given properties. Note that much more efficient from this

point of view is the combination of the symmetry reduction with the methods of qualitative analysis.

Aside from the symmetry reduction, there exist another group of methods enabling to obtain exact solutions to nonlinear PDE's. They are based on choosing the proper transformation (or ansatz), simplifying the problem. As an example let us mention the famous Cole-Hopf transformation, which was originally used for the non-local linearization of Burgers equation. Later on employment of the similar transformations enabled Hirota to obtain the multi-soliton solutions of the completely integrable KdV equation without referring to the inverse scattering method [1]. An intense development of the ansatz-based method in the following years was highly simulated by the fact that it proved to be effective for obtaining the particular exact solutions of the evolutionary PDE's that are not completely integrable. The essence of the ansatz-based method is well summarized in the recent work of E. Fan [3]. Beside the imposing bibliography, it contains general formulation of the unified algebraic method, which will be presented in next section. Literally during last years the approach presented by E.Fan was used in the composition with the generalized Cole-Hopf anzatz, which made possible to obtain a series of exact solutions of nonlinear transport equation [4, 5].

In this paper we present certain modification of the anzatz-based method presented in [4]. In accordance with the core of our interests, we use it to obtain exact solutions of hyperbolic modification of the non-linear transport equation, which takes into account the nonlocal effects. We concentrate on finding out the particular types of travelling wave (TW) solutions, namely, the soliton-like and kink-like solutions, but this approach can be easily used for searching out any exact solution which can be described as an algebraic combination of certain types of special functions. Next we consider the family of TW solutions for the spatially nonlocal hydrodynamic-type model. Using the qualitative analysis we state the existence of periodic and soliton-like TW solutions. Imposing some restrictions on the parameters we obtain the soliton-like solution in the analytical form. It turns out to be much more complicated than that obtained within the above mentioned anzatz-based methods.

2 Exact TW solutions to the hyperbolic modification of nonlinear transport equations

2.1 Unified algebraic method and it's modifications

The essence of the anzatz-based method or the *direct unified algebraic method* in E.Fan's terminology [3], is based on the observation that the particular solutions of any system of PDE's

$$H^{\nu}(u^i,\ u^i_t,\ u^i_x,\ u^i_{xx},...) \qquad i=1,2,...m, \quad \nu=1,2,..,n \tag{1}$$

which do not depend on (t,x) coordinates in explicit form, can be presented as a linear combination

$$u^i=\sum_{\mu=0}^{n}a^i_{\mu}\phi^{\mu}(\xi) \qquad \xi=x+vt, \tag{2}$$

where a_{μ} are unknown parameter, while function $\phi(\xi)$ - satisfies the equation

$$\phi(\xi)=\pm\sqrt{\sum_{\nu=0}^{r}c_{\nu}\phi^{\nu}}. \tag{3}$$

Depending on the conditions posed on the parameters c_{ν}, solutions of the equation (3) are expressed by the elliptic Jacobi (Weierstrass), hyperbolic or trigonometric functions [3]. The properties of these functions are inherited by those solution of the initial system that can be presented in the form (2). In fact this methodology is constructive and algorythmic if the functions H^{ν} arising in (1) are algebraic ones. One easily gets convinced that with this assumptions the substitution of (2) into the initial system (1) gives the polynomial functions with respect to $\phi^{\mu}(\xi)$ and $\phi^{\mu}\,\dot{\phi}(\xi)$. Equating to zero the coefficients standing at the corresponding powers of these functions, we obtain nonlinear system of the algebraic equations which determines particular solutions of the initial system.

The enforced version of the Fan's method was put forward recently in [4, 5]. It was used for searching out the exact solutions of the equation

$$u_t+Auu_x-\kappa u_{xx}=f(u). \tag{4}$$

More preciesly, there has been proposed the ansatz

$$u=\left[\frac{z'(\xi)}{z(\xi)}\right]^k, \qquad \xi=x+vt+x_0, \qquad k=1,2,.., \tag{5}$$

where $z = \sum a_\mu \phi^\mu(\xi)$, and $\phi(\xi)$ is the function satisfying the (3). Owing to this combination, the multi-parameter families of the exact solutions were obtained in the situation when the pure Fan's methodology does not work [4].

On analyzing different versions of the anzatz-based method, one can conclude that effectiveness of their employment is based on the mere fact that the family of functions $\phi^\mu(\xi)$ and $\phi^\nu \dot{\phi}(\xi)$ is closed with respect to the algebraic operations and differentiating. In view of this a quite natural generalization of the already mention ansatze would be as follows:

$$u = \frac{f(\xi)}{g(\xi)} = \frac{\sum_{\mu=0}^{m_1} a_\mu \phi^\mu(\xi) + \dot{\phi(\xi)} \sum_{\nu=0}^{m_2} b_\nu \phi^\nu(\xi)}{\sum_{\lambda=0}^{n_1} c_\lambda \phi^\lambda(\xi) + \dot{\phi(\xi)} \sum_{\kappa=0}^{n_2} d_\kappa \phi^\kappa(\xi)}, \tag{6}$$

where the function $\phi(\xi)$ still satisfies the equation (3), but, in contrast to (5), dependence between the functions f and g is not assumed from the very beginning. Effectiveness of the anzatz (6) is demonstrated in the following subsection.

2.2 Exact travelling wave solution to the nonlinear hyperbolic equation

Let us consider the following equation:

$$\tau u_{tt} + A u u_x + B u_t - \kappa u_{xx} = f(u) = \sum_{\nu \in I} \lambda_\nu u^\nu, \tag{7}$$

where τ, A, B, κ are non-negative constants. For $A = 0$ equation (7) coincides with the nonlinear telegraph equation; for $A \neq 0$ it coincides with the hyperbolic generalization of Burgers equation, while for $A = B = 0$ – with the nonlinear d'Alambert equation.The hyperbolic modifications of nonlinear transport equations arise in a natural way when the memory effects are taken into account [6].

The main goals of this paper are to obtain the exact soliton-like and kink-like solutions of equation (7) and to present the advantages of anzatz (6) compared to the already mentioned modification [4] of the unified algebraic method.

Assuming that the solitons and kinks can be expressed by powers of function $sech(\xi)$, which is the particular solution of equation (3) and function $sinh(\xi)$ which appears in the odd derivatives of the function $sech(\xi)$, we use the following ansatz:

$$u(\xi) = \frac{f(\xi)}{g(\xi)} = \frac{\sum_{\mu=0}^{m_1} a_\mu sech^\mu(\alpha\xi) + sh(\alpha\xi) \sum_{\nu=0}^{m_2} b_\nu sech^\nu(\alpha\xi)}{\sum_{\gamma=0}^{n_1} c_\gamma sech^\gamma(\alpha\xi) + sh(\alpha\xi) \sum_{\sigma=0}^{n_2} d_\sigma sech^\sigma(\alpha\xi)} \tag{8}$$

or, what is the same,

$$u(\xi) = \frac{\sum_{\mu=0}^{m} a_\mu exp(\mu\alpha\xi)}{\sum_{\nu=0}^{n} b_\nu exp(\nu\alpha\xi)}. \tag{9}$$

Inserting anzatz (8) (or (9)) into (7) and executing of all necessary operations, we obtain an algebraic equation containing, respectively, functions $sech^\mu(\alpha\xi)$, $sech^\nu(\alpha\xi)sh(\alpha\xi)$ or $\exp[\mu\,\alpha\,\xi]$. Regarding them as the functionally independent ones and equating to zero corresponding coefficients, we go to the system of algebraic equations. We do not expose the details of these calculations since they are simple but cumbersome. To accomplish them we used the package of symbolic computation "Mathematica". The results obtained are presented below.

I. For arbitrary $A,\ B$ and $f(u) = \lambda_0 + \lambda_1 u(t,x) + \lambda_2 u(t,x)^2 + \lambda_3 u(t,x)^3$ function

$$u(t,x) = \frac{a_0 + a_1 e^{\alpha(x+vt)}}{b_0 + b_1 e^{\alpha(x+vt)}} \tag{10}$$

satisfies equation (7) if the following conditions hold:

$$\begin{gathered} \lambda_0 = \tfrac{-a_0 a_1 \alpha}{\Delta^2}(Bv\Delta + h\Theta), \quad \lambda_1 = \tfrac{1}{b_0 b_1 \Delta^2}\left[\alpha b_0 b_1 (Bv\Theta\Delta + h\Theta^2) + \lambda_3 a_0 a_1 \Delta^2\right], \\ \lambda_2 = \tfrac{-1}{b_0 b_1 \Delta^2}\left[\alpha b_0^2 b_1^2 (Bv\Delta + h\Theta) + \lambda_3 \Delta^2 \Theta\right], \quad A = \tfrac{-1}{\alpha b_0 b_1 \Delta}\left[-2h\alpha b_0^2 b_1^2 + \lambda_3 \Delta^2\right]. \end{gathered} \tag{11}$$

Here and henceforth we use the notation $h = \alpha(v^2\tau - \kappa)$, $\Delta = a_1 b_0 - a_0 b_1$, $\Theta = a_1 b_0 + a_0 b_1$. Equation (10) defines a kink-like regime when $b_0 b_2 > 0$ and $a_0/b_0 \neq a_2/b_2$. Using the conditions (11), we can express the unknown parameters from formula (10) by the parameters characterizing equation (7), yet, in general case it is too cumbersome. It is much easy to do when $\alpha = 2\sqrt{-\lambda_0\lambda_2}/v$, $a_0 = -a_1 = \sqrt{-\lambda_0/\lambda_2}$, $b_0 = b_1 = 1$. From these conditions we obtain the solution:

$$u(t,\,x) = \sqrt{-\lambda_0/\lambda_2}\,Tanh\left[\frac{\sqrt{-\lambda_0\lambda_2}}{v}(x+vt)\right]$$

and conditions: $v = \frac{\lambda_2 (A\,B + \sqrt{A^2\,B^2 - 8\kappa\,\lambda_3 + 16\kappa\,\lambda_2^2\,\tau})}{2\lambda_3 - 4\lambda_2^2\,\tau}$, $\lambda_1 = \frac{\lambda_0\,\lambda_3}{\lambda_2}$. For $\kappa = 1$, $\tau = 0$, $A = 0$, $B = 1$ function (10) coincides with the solution obtained in [4].

Another example of the kink-like solution defined by the formulae (10)-(11) is as follows:

$$u(t,\,x) = \frac{2}{b_0[1 + \exp(2\,\alpha\,\xi)]}, \qquad \xi = x + vt,$$

where $b_0 = (-\lambda_2 \pm \sqrt{\lambda_2^2 - 4\,\lambda_1\,\lambda_3})/\lambda_1,\ A = \lambda_0 = 0,\ \alpha = -(2\lambda_2 + 3\,b_0\,\lambda_1)/[4B\,v\,b_0]$, and

$$v = \frac{\pm\sqrt{\kappa}(2\lambda_2 + 3\,b_0\,\lambda_1)}{\sqrt{4\lambda_2^2\tau + 4b_0\lambda_2(B^2 + 3\lambda_1\tau) + b_0^2\lambda_1(2B^2 + 9\lambda_1\tau)}}.$$

II. For $A = 0$, arbitrary B and $f(u) = \lambda_0 + \lambda_{1/2}u(t,x)^{\frac{1}{2}} + \lambda_1 u(t,x) + \lambda_{3/2}u(t,x)^{\frac{3}{2}} + \lambda_2 u(t,x)^2$ function

$$u(t,x) = \left[\frac{a_0 + a_1 e^{\alpha(x+vt)}}{b_0 + b_1 e^{\alpha(x+vt)}}\right]^2 \tag{12}$$

satisfies the equation(7) providing that the following conditions hold:

$$\lambda_0 = 2a_0^2 a_1^2 \alpha h/\Delta^2, \quad \lambda_{1/2} = -2a_0 a_1 \alpha(3h\Theta + Bv\Delta)/\Delta^2,$$

$$\lambda_1 = 2\alpha(h(3\Theta^2 - \Delta^2) + Bv\Delta\Theta)/\Delta^2, \quad \lambda_{3/2} = -2b_0 b_1 \alpha(5h\Theta + Bv\Delta)/\Delta^2,$$

$$\lambda_2 = 6b_0^2 b_1^2 h\alpha/\Delta^2.$$

This solution defines the solitary wave regime if $b_0 b_1 > 0$, $|a_0|/|b_0| = |a_2|/|b_2|$ while $a_0/b_0 \neq a_2/b_2$.

III. For $B = 0$, arbitrary A and $f(u) = \lambda_1 u(t,x) + \lambda_3 u(t,x)$ function

$$u(t,x) = \frac{a_1 e^{\alpha\xi} + a_2 e^{2\alpha\xi}}{-a_1^3 - 3\,a_1^2\,a_2\,e^{\alpha\xi} + 3\,a_1\,a_2^2\,e^{2\alpha\xi} + a_3^3\,e^{3\alpha\xi}}, \qquad \xi = x + vt \tag{13}$$

satisfies (7), when $\lambda_1,\ \lambda_3$ are positive and the parameters are as follows: $6\,a_1\,a_2 = -\sqrt{\lambda_3/\lambda_1}$, $\alpha = \sqrt{\lambda_1\,\lambda_3}/A,\ \ v = \pm\sqrt{(A^2/\lambda_3 + \kappa)/\tau}$. This solution is always singular, because for arbitrary values of the parameters the expression in the denominator of the formula (13) nullifies for some $\xi \in R^1$.

IV. Now let us consider the case $A = B = 0$.

IV a. For $f(u) = \lambda_0 + \lambda_1 u(t,x) + \lambda_2 u(t,x)^2 + \lambda_3 u(t,x)^3$ function

$$u(t,x) = \frac{a_0 + 2a_1 e^{\alpha(x+vt)} + a_0 e^{2\alpha(x+vt)}}{b_0 + 2b_1 e^{\alpha(x+vt)} + b_0 e^{2\alpha(x+vt)}}, \tag{14}$$

satisfies equation (7) when the following conditions hold:

$$\begin{aligned}
\lambda_0 &= a_0\,(2a_0^2\,b_0 - a_1^2\,b_0 - a_0\,a_1\,b_1\,)\alpha\,h/\Delta^2 \\
\lambda_1 &= ((a_1^2\,b_0^2 + 4a_0\,a_1\,b_0\,b_1 + a_0^2\,(-6\,b_0^2 + b_1^2))\alpha\,h/\Delta^2 \\
\lambda_2 &= 3\,b_0\,(2\,a_0\,b_0^2 - a_1\,b_0\,b_1 - a_0\,b_1^2)\alpha\,h/\Delta^2 \\
\lambda_3 &= -2\,b_0\,(b_0^2 - b_1^2)\alpha\,h/\Delta^2
\end{aligned} \tag{15}$$

For $a_0 \neq 0$, $b_0 \neq 0$, $|a_1| + |b_1| \neq 0$ equation (14) defines the soliton-like solution. One of the parameters, contained in (14) can be chosen arbitrarily, while the rest can be expressed, using the (15), as the functions of the parameters, defining equation (7). We omit doing this in the general case, but present one particular example. Thus, for $b_1 = 0$, $b_0 = 1$ and arbitrary α we have the solution

$$u(t,\, x) = \frac{a_0 + 2a_1 e^{\alpha(x+vt)} + a_0 e^{2\alpha(x+vt)}}{1 + e^{2\alpha(x+vt)}},$$

with $a_0 = -\lambda_2/(3\,\lambda_3)$, $a_1 = \sqrt{2(\lambda_2^2 - \lambda_1\,\lambda_3)/\lambda_3^2}$, $v = \pm\sqrt{\left[\lambda_1 - \lambda_2^2/(3\,\lambda_3) + \kappa\,\alpha^2\right]/(\tau\,\alpha^2)}$, with the additional condition $\lambda_0 + \lambda_1 a_0 + \lambda_2 a_0^2 + \lambda_3 a_0^3 = 0$.

VI b. For $f(u) = \lambda_{1/2} u(t,x)^{1/2} + \lambda_1 u(t,x) + \lambda_{3/2} u(t,x)^{3/2} + \lambda_2 u(t,x)^2$ we obtain the soliton-like solution

$$u(t,x) = \frac{(e^{\alpha(x+vt)} + 1)^4}{(b_0 e^{2\alpha(x+vt)} + (2b_0 + 4b_1)e^{\alpha(x+vt)} + b_0)^2} \tag{16}$$

with

$$\begin{aligned} \lambda_{1/2} &= -3\,\alpha\,h/b_1 & \lambda_1 &= (12\,b_0 + 4\,b_1)\alpha\,h/b_1 \\ \lambda_{3/2} &= -(15\,b_0^2 + 10\,b_0\,b_1)\alpha\,h/b_1 & \lambda_2 &= (6\,b_0^2\,b_1 + 6\,b_0^3)\alpha\,h/b_1. \end{aligned}$$

VI c. For $f(u) = \lambda_0 + \lambda_{1/2} u(t,x)^{\frac{1}{2}} + \lambda_1 u(t,x) + \lambda_{3/2} u(t,x)^{\frac{3}{2}}$ the localized wave pack

$$u(t,\, x) = \frac{(a_0 e^{2\alpha(x+vt)} + (2a_0 + 4a_1)e^{\alpha(x+vt)} + a_0)^2}{(e^{\alpha(x+vt)} + 1)^4} \tag{17}$$

defines a solution of (7) if the following conditions hold:

$$\begin{aligned} \lambda_0 &= 2a_0^2(a_0 + a_1)\,\alpha\,h/a_1 & \lambda_{1/2} &= (9a_0^2 + 6a_0 a_1)\,\alpha\,h/a_1 \\ \lambda_1 &= (12a_0 + 4a_1)\,\alpha\,h/a_1 & \lambda_{3/2} &= -5\,\alpha\,h/a_1. \end{aligned}$$

VI d. For $f(u) = \lambda_1 u(t,x) + \lambda_{3/2} u(t,x)^{\frac{3}{2}} + \lambda_2 u(t,x)^2$ function

$$u(t,x) = \frac{4e^{2\alpha(x+vt)}}{(a_0 e^{2\alpha(x+vt)} + 2a_1 e^{\alpha(x+vt)} + a_0)^2} \tag{18}$$

with

$$\begin{aligned} \lambda_1 &= 4\,\alpha\,h & \lambda_{3/2} &= -10\,a_1\alpha\,h \\ \lambda_2 &= (6\,a_1^2 - 6\,a_0^2)\alpha\,h. \end{aligned}$$

defines the soliton-like solution of the equation (7).

VI e. Finally, let us consider equation

$$\tau u_{tt} - \kappa u_{xx} = \lambda_0 + \lambda_1 u + \lambda_2 u^2 + \lambda_3 u^3. \tag{19}$$

Inserting the anzatz $u = \phi(\xi), \quad \xi = x + v\,t$ into the equation (19), we obtain, after one integration, the following ODE:

$$\frac{d\phi}{d\xi} = \pm\sqrt{c_0 + c_1\,u + c_2\,u^2 + c_3\,u^3 + c_4\,u^4}, \tag{20}$$

where c_0 is an arbitrary constant, $c_1 = 2\lambda_0/H$, $c_2 = \lambda_1/H$, $c_3 = 2\lambda_2/(3\,H)$, $c_4 = \lambda_3/(2\,H)$, $H = \tau\,v^2 - \kappa$. To this equation the classification given in [3] is applied:

(a) if $\lambda_2 = \lambda_0 = 0$, then equation (19) possesses a soliton-like solution

$$u(t,\,x) = \sqrt{-2\lambda_1/\lambda_3} sech(\sqrt{\lambda_1/H}\xi) \qquad \lambda_1 > 0, \qquad \lambda_3 < 0; \tag{21}$$

(b) if $\lambda_0 = \lambda_2 = 0$, then equation (19) posesses a kink-like solution

$$u(t,\,x) = \sqrt{-\lambda_1/\lambda_3}\,\tanh\left[\sqrt{-\lambda_1}/(2\,H)\,\xi\right], \qquad \lambda_1 < 0, \qquad \lambda_3 > 0; \tag{22}$$

(c) if $\lambda_0 = \lambda_3 = 0$, then equation (19) possesses a soliton-like solution

$$u(t,\,x) = -\,[3\lambda_1/(2\lambda_2)]\,sech^2\left[\sqrt{\lambda_1/H}\,\xi/2\right] \qquad \lambda_1 > 0. \tag{23}$$

Presented above results enable us to state that the anzatze (8) and (9) are effective and their employment gives the exact solutions in the situations when the ansatze suggested in [4, 3] do not work. Note, that as a by-product we obtained a number of new exact solutions of non-linear transport and Burgers equations. These solutions can be easily extracted from the presented above formulae by simple substitution $B = 1, \tau = 0$ (and also $A = 0$ when it is necessary). Thus, the proposed modification of the unified algebraic method proves to be useful and its employment results in essential broadening the number of solutions of the given type (i.e. soliton-like and kink-like solutions), which can be obtained in analytic form. Yet, as it will be shown in the following section, none of the version of the anzatz-based method proposed by now is fully universal.

3 Periodic and soliton-like TW solutions of the nonlocal hydrodynamic-type model.

In conclusion, let's analyze the family of TW solutions for the following system:

$$\begin{aligned} &u_t + \beta \rho^{\nu+1} \rho_x + \sigma \left[\rho^{\nu+1} \rho_{xxx} + 3(\nu+1) \rho^{\nu} \rho_x \rho_{xx} + \nu(\nu+1) \rho^{\nu+1} \rho^{\nu-1} \rho_x^3 \right] = 0, \\ &\rho_t + \rho^2 u_x = 0, \end{aligned} \tag{24}$$

where ν, β, σ are constants. The system (24) arises in a natural when the balance equations for mass and momentum, taken in the hydrodynamic approximation, are closed by the dynamic equation of state, accounting for the short-ranged spatial non-locality [7]. In general case the answer on the existence of the periodic and soliton-like TW solutions is obtained by the methods of qualitative analysis, but under some additional conditions posed on the parameters we are able to present some of them in the analytic form, omitting the anzatz-based method (which doesn't work in this case).

Let us consider the following family of invariant travelling wave solutions:

$$u = U(\omega), \qquad \rho = R(\omega), \qquad \omega = x - Dt. \tag{25}$$

Inserting the anzatz (25) into the second equation of system (24), we obtain the first integral $U = C_1 - D/R$ and the system of ODE's

$$\begin{cases} \frac{dR}{d\omega} = Y \\ \frac{dY}{d\omega} = (\sigma R^{\nu+2})^{-1} \left\{ ER - [D^2 + \beta R^{\nu+3}/(\nu+2) + \sigma(\nu+1) R^{\nu+1} Y^2] \right\}. \end{cases} \tag{26}$$

In accordance with the asymptotic conditions $\lim_{\omega \to +\infty} U(\omega) = 0$, $\lim_{\omega \to +\infty} R(\omega) = R_1 > 0$, $U(+\infty) = 0$, $R(+\infty) = R_1 > 0$, we assume henceforth that $C_1 = D/R_1$ and $E = D^2/R_1 + \beta\, R_1^{\nu+2}/(\nu+2)$.

Dividing the second equation of system (26) by the first one and introducing new variable $Z = Y^2 \equiv (d\,R/d\,\omega)^2$, we get, after some algebraic manipulation, the linear inhomogeneous equation

$$Z'(R) + 2[(\nu+1)\,R]^{-1} Z(R) = 2\left[E\,R - D^2 - \beta R^{\nu+3}/(\nu+2)\right]/(\sigma\, R^{\nu+2}). \tag{27}$$

Solving this equation with respect to $Z = Z(R)$ and next integrating the equation obtained after the substitution $Z = (d\,R/d\,\omega)^2$, we can express the solution of (26) as the following quadrature:

$$\omega - \omega_0 = \int \frac{\pm\sqrt{\sigma}\, R^{1+\nu}\, d\,R}{\sqrt{H_1 + 2\,E\,\frac{R^{2+\nu}}{(2+\nu)} - 2\,D^2 \frac{R^{1+\nu}}{(1+\nu)} - \beta\,\frac{R^{2(2+\nu)}}{(2+\nu)^2}}}. \tag{28}$$

Unfortunately, the direct analysis of solution (28) is very difficult. To realize what sort of solutions we deal with, the qualitative analysis of system (28) is performed. It is evident, that all isolated critical points of system (26) are located on the (horizontal) axis OR. They are determined by solutions of the algebraic equation $P(R) = \beta\, R^{\nu+3}/(\nu+2) - ER + D^2 = 0$. One of the roots of this equation coincides with R_1. Location of the second real root depends on relations between the parameters. If $\nu > -2$ and $D^2 > \beta R_1^{\nu+3}$, then there exists the second critical point $A_2(R_2,\, 0)$ with $R_2 > R_1$ and the polynomial $P(R)$ has the representation

$$P(R) = (R - R_1)(R - R_2)\Psi(R). \tag{29}$$

For any $\nu > -2$ function $\Psi(R)$ is positive whenever $R > 0$, since the function $\beta\, R^{\nu+3}/(\nu+2)$ is concave in this interval and has exactly two intersections with the line $E\,R - D^2$.

Analysis of system's (26) linearization matrix shows, that the critical points $A_1(R_1, 0)$ is a saddle, while the critical point $A_2(R_2, 0)$ is a center. Thus, system (26) has only such critical points, that are characteristic to the hamiltonian system. This circumstance suggests that there could exist a hamiltonian system equivalent to (26) In fact, the following statement holds.

Lemma. *If to introduce a new independent variable T, obeying the equation $d\,\omega/dT = 2\,\sigma R^{2(\nu+1)}$, then system (26) can be written as a hamiltonian one*

$$\begin{cases} \frac{dR}{dT} = 2\,\sigma R^{2(\nu+1)} \equiv \frac{\partial H}{\partial Y}, \\ \frac{dY}{dT} = 2\,R^{\nu}\left(ER - [D^2 + \beta\, R^{\nu+3}/(\nu+2) + \sigma(\nu+1)R^{\nu+1}Y^2]\right) \equiv -\frac{\partial H}{\partial R}, \end{cases} \tag{30}$$

with

$$H = 2D^2\frac{R^{\nu+1}}{(\nu+1)} + \beta\frac{R^{2(\nu+2)}}{(\nu+2)^2} + \sigma Y^2 R^{2(\nu+1)} - 2E\frac{R^{\nu+2}}{(\nu+2)}. \tag{31}$$

By elementary checking one can get convinced that the function H is constant on phase trajectories of both systems (26) and (30), and since the integrating multiplier $\Psi = 2R^{\nu}$, occurring in formula (30) is positive for $R > 0$, then phase trajectories of systems (26) and (30) are similar in the right half-plane of the phase space $(R,\, Y)$. Thus all the statements concerning the geometry of the phase trajectories of system (30) lying in the right half-plane are applicable to corresponding solutions of system (26). In particular, we immediately conclude that the critical point $A_2(R_2,\ 0)$ remains a center when the nonlinear terms are added. This means that the initial system (24) possesses a one-parameter family of periodic solutions. If the right branches of the separatrices of the saddle $A_1(R_1,\ 0)$ go to infinity (the stable branch W^s when $t \to -\infty$ and the unstable branch W^u when $t \to +\infty$), then the

domain of finite periodic motions is unlimited. Another possibility is connected with the existence of the homoclinic trajectories. In this case the initial system possesses localized soliton-like regimes. To answer the question on which of the above mentioned possibilities is realized in system (26), the behavior of the saddle separatices, lying to the right from the line $R = R_1$, should be analyzed. We obtain the equation for saddle separatices by putting $H = H(R_1, 0) = H_1$ in the LHS of the equation (31) and solving it next with respect to Y:

$$Y = \pm\sqrt{\frac{H_1 + 2ER^{\nu+2}/(\nu+2) - [2D^2\,R^{\nu+1}/(\nu+1) + \beta R^{2(\nu+2)}/(\nu+2)^2]}{\sigma R^{2(\nu+1)}}} \tag{32}$$

It is evident from equation (32), that incoming and outgoing separatrices are symmetrical with respect to OR axis. Therefore we can restrict our analysis to one of them, e.g. to the upper separatrix Y_+. First of all, let us note, that in the point$(R_1,\ 0)$ separatrix Y_+ forms with OR axis a positive angle $\alpha = arctg\sqrt{(R_2 - R_1)\Psi(R_1)/\left(\sigma R_1^{\nu+2}\right)}$. The above formula arises from the linear analysis of system (26) in critical point $A_1(R_1,\ 0)$. So $Y_+(R)$ is increasing when $R - R_1$ is small and positive. On the other hand, function $G(R) = H_1 + 2E\,R^{\nu+2}/(\nu+2) - \left[2D^2\,R^{\nu+1}/(\nu+1) + \beta\,R^{2(\nu+2)}/(\nu+2)^2\right]$, standing inside the square root in equation (32), tends to $-\infty$ as $R \to +\infty$, because the coefficient at the highest order monomial $R^{2(\nu+2)}$ is negative, while the index $\nu + 2$ is assumed to be positive. Therefore the function $G(R)$ intersects the open set $R > R_1$ of the OR axis at least once. Let us denote a point of the first intersection by $A_3(R_3, 0)$. The coordinate of intersection satisfies inequality $R_3 > R_2$. It can be easily seen by noting that $G'(R) = -2R^\nu\left(\beta R^{\nu+3}/(\nu+2) - ER + D^2\right) = -2R^\nu P(R)$ and analyzing $\lim_{R\to R_3^-} dY/d\,R$. On the other hand, $G'(R) < 0$ when $R > R_2$, and this is suffice to show that $\lim_{R\to R_3^-} dY/d\,R = -\infty$. Due to the symmetry of the saddle separatrices, they intersect tangently in the point $A_3(R_3,\ 0)$, The result obtained can be formulated as follows.

Theorem *If $\nu > -2$ and $D^2 > \beta\,R_1^{\nu+3}$, then system (26) possesses a one parameter family of periodic solutions and the homoclinic solution, formed by the tangent intersection of separatrices of the saddle point $A_1\,(R_1,\ 0)$.*

Thus we have shown, that system (24) possesses periodic and soliton-like invariant solutions. Let us note in conclusion that for some special case the integral standing at the RHS of the formula (28) can be calculated explicitely:

$$\omega = \pm\left\{\sqrt{8}\arcsin\left[\frac{R+1}{2\sqrt{2}}\right] + \sqrt{2}\,\log\left[\frac{R-1}{3-R+\sqrt{20-2(R^2+2R+3)}}\right] - \omega_0\right\}, \tag{33}$$

where $\omega_0 = \sqrt{2}\,\pi - \log 2$. This solution corresponds to the following values of the parameters: $D = 1 = R_1 = \sigma = 1$, $\beta = 1/2$, $\nu = 0$, and $E = 5/4$.

4 Conclusions.

Thus we presented the effective anzatz resulting from the analysis of the methods put forward in [3, 5, 4] and based on the following observation: the crucial element of any of anzatz-based method is the closeness of certain class of functions with respect to the algebraic operations and differentiating. This observation served as the main motive when we put forward the anzatz (6). Note, that it can be easily modified for obtaining the periodic solutions or, more generally, exact solutions that can be expressed as algebraic combinations of some special functions.

The results of the last section show that the combination of the self-similar reduction and the qualitative analysis can deliver the exhaustive answer to the question on the existence of certain types of solutions within the given family. At the end of this section we also present the exact homoclinic solution, that was predicted earlier by the qualitative analysis but failed to be obtained within the anzatz-based methods. The reason of this is evidently linked with the fact that solution (33) is too complicated to be expressed as an algebraic combination of the hyperbolic functions. Whether it can be expressed in terms of special functions is still an open problem, connected with the more general problem of finding out a fully universal ansatz.

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